

# On asymptotics of large Haar distributed unitary matrices<sup>1</sup>

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Let  $U_n$  be an  $n \times n$  Haar unitary matrix. In this paper, the asymptotic normality and independence of  $\text{Tr } U_n, \text{Tr } U_n^2, \dots, \text{Tr } U_n^k$  are shown by using elementary methods. More generally, it is shown that the renormalized truncated Haar unitaries converge to a Gaussian random matrix in distribution.

*Key words: random matrices, joint eigenvalue distribution, Haar unitary, truncated Haar unitary, method of moments, trace of powers.*

## 1 Introduction

Entries of a random matrix are random variables but a random matrix is equivalently considered as a probability measure on the set of matrices. A simple example of random matrix has independent identically distributed entries. In this paper random unitary matrices are studied whose entries must be correlated.

A unitary matrix  $U = (U_{ij})$  is a matrix with complex entries and  $UU^* = U^*U = I$ . In terms the entries these relations mean that

$$\sum_{j=1}^n |U_{ij}|^2 = \sum_{i=1}^n |U_{ij}|^2 = 1, \text{ for all } 1 \leq i, j \leq n, \quad (1)$$

$$\sum_{l=1}^n U_{il} \bar{U}_{jl}, \text{ for all } 1 \leq i, j \leq n, \quad i \neq j. \quad (2)$$

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The set  $\mathcal{U}(n)$  of  $n \times n$  unitary matrices forms a compact topological group with respect to the matrix multiplication and the usual topology, therefore there exists a unique (up to the scalar multiplication) translation invariant measure on  $\mathcal{U}(n)$ , the so-called Haar measure. We will consider a random variable  $U_n$  which maps from a probability space to  $\mathcal{U}(n)$ , and take its values uniformly from  $\mathcal{U}(n)$ , i.e. if  $H \subset \mathcal{U}(n)$ , then

$$\text{Prob}(U_n \in H) = \gamma(H),$$

where  $\gamma$  is the normalized Haar measure on  $\mathcal{U}(n)$ . We call this random variable a Haar unitary random variable, or shortly Haar unitary.

Relations (1) and (2) show that the column vectors of a unitary are pairwise orthogonal unit vectors and the distribution of each column has unitarily invariant joint distribution in case of a Haar distributed unitary. This fact allows a simple construction of a Haar unitary from a Gaussian matrix with i.i.d. entries. In Sect. 2 this construction is written up in details. The construction allows to determine the distribution of the matrix elements. Due to the permutation invariance, the elements are identically distributed. We observe that large powers of the eigenvalues are independent and uniformly distributed. The main aim of the paper is to study asymptotic questions when the matrix size is going to infinity. In this limit the matrix elements are going to be Gaussian (after a renormalization) and more generally, the truncated matrix converges to a Gaussian matrix. This is the content of Sect. 3. In the rest of the paper we show that the trace of the powers is going to Gaussian in the limit, moreover the traces of different powers are asymptotically independent. Actually, this has been shown by Diaconis and Shahshahani [3], they determined the Fourier transform of the limit distribution of the eigenvalues. In their proof the characters of the symmetric and unitary groups, and different bases of the symmetric polynomials play the main role. Here we get the same result in a more elementary way. The method of moments is used, and we examine the order of magnitude of the different summands in the trace. This method could be familiar from Arnold [1] when he studied Wigner matrices, or from Bai and Yin [2] for sample covariance matrices.

Random unitary matrices may be applied to several physical phenomena, such as chaotic scattering or statistical properties of periodically driven quantum systems [5]. The large deviation theorem for the empirical eigenvalue density of Haar unitaries (and of some other unitaries) was established in [7]. Independent Haar unitaries provide also a simple example of asymptotical freeness [6, 8].

## 2 Haar unitary matrices

Let  $\xi$  be a complex-valued random variable. If  $\text{Re } \xi$  and  $\text{Im } \xi$  are independent and normally distributed according to  $N(0, 1/2)$ , then we call  $\xi$  a *standard complex normal variable*. The terminology is justified by the properties  $E(\xi) = 0$  and  $E(\xi\bar{\xi}) = 1$ .

**Lemma 2.1** Assume that  $R \geq 0$  and  $R^2$  has exponential distribution with parameter 1,  $\theta$  is uniform on the interval  $[0, 2\pi]$ , and assume that  $R$  and  $\theta$  are independent. Then  $\xi = Re^{i\theta}$  is a standard complex normal random variable and

$$E(\xi^k \bar{\xi}^\ell) = \delta_{k\ell} k! \quad (k, \ell \in \mathbb{Z}_+)$$

*Proof.* Let  $X$  and  $Y$  be real-valued random variables and assume that  $X + iY$  is standard complex normal. For  $r > 0$  and  $0 \leq \theta_0 \leq 2\pi$  set

$$S_{r, \theta_0} := \{\rho e^{i\psi} : 0 \leq \rho \leq r, 0 \leq \psi \leq \theta_0\},$$

then

$$\begin{aligned} P(X + iY \in S_{r, \theta_0}) &= \frac{1}{\pi} \int \int_{\{(s, t) : s + it \in S_{r, \theta_0}\}} e^{-(s^2 + t^2)} ds dt \\ &= \frac{1}{\pi} \int_0^{\theta_0} d\psi \int_0^r \rho e^{-\rho^2} d\rho \\ &= \frac{1}{2\pi} \theta_0 (1 - e^{-r^2}) = P(\xi \in S_{r, \theta_0}). \end{aligned}$$

This proves the first part which makes easy to compute the moments:

$$E(\xi^k \bar{\xi}^\ell) = E(R^{k+\ell}) E(e^{i\theta(k-\ell)}) = \delta_{k\ell} E(R^{2k}),$$

so we need the moments of the exponential distribution. We have by partial integration

$$\begin{aligned} \int_0^\infty x^k e^{-x} dx &= -[x^k e^{-x}]_0^\infty + k \int_0^\infty x^{k-1} e^{-x} dx = k \int_0^\infty x^{k-1} e^{-x} dx \\ &= k(k-1) \int_0^\infty x^{k-2} e^{-x} dx = \dots = k! \int_0^\infty e^{-x} dx = k! \end{aligned}$$

which completes the proof of the lemma.  $\square$

Next we recall how to get a Haar unitary from a Gaussian matrix with independent entries by the Gram-Schmidt orthogonalization procedure on the column vectors.

Suppose that we have a complex random matrix  $Z$  whose entries  $Z_{ij}$  are mutually independent standard complex normal random variables. We perform the Gram-Schmidt orthogonalization procedure on the column vectors  $Z_i$  ( $i = 1, 2, \dots, n$ ), i.e.

$$U_i = \frac{Z_i - \sum_{l=1}^{i-1} \langle Z_i, U_l \rangle U_l}{\left\| Z_i - \sum_{l=1}^{i-1} \langle Z_i, U_l \rangle U_l \right\|}, \quad (3)$$

where

$$\|(X_1, X_2, \dots, X_n)\| = \sqrt{\sum_{k=1}^n |X_k|^2}.$$

**Lemma 2.2** *The above column vectors  $U_i$  constitute a unitary matrix  $U = (U_i)_{i=1,\dots,n}$ . Moreover, for all  $V \in \mathcal{U}(n)$  the distributions of  $U$  and  $VU$  are the same.*

*Proof.* First we prove, that for any  $V \in \mathcal{U}(n)$  the matrices  $Z$  and  $VZ$  have the same distribution. The entries  $\xi_{ij}$  of  $VZ$  are standard complex normal. Indeed,

$$\xi_{ij} = \sum_{l=1}^n V_{il} Z_{lj}$$

is normal. Furthermore

$$E(\xi_{ij}) = \sum_{l=1}^n V_{il} E(Z_{lj}) = 0,$$

and

$$E(\xi_{ij} \bar{\xi}_{ij}) = \sum_{l=1}^n |V_{il}|^2 E(Z_{lj} \bar{Z}_{lj}) = \sum_{l=1}^n |V_{il}|^2 = 1.$$

Next we prove that the correlation between two entries is zero. (In the case of normally distributed random variables this is equivalent to the independence.)

$$\begin{aligned} E(\xi_{ij} \bar{\xi}_{sr}) &= E \left( \left( \sum_{l=1}^n V_{il} Z_{lj} \right) \left( \sum_{k=1}^n \overline{V_{sk} Z_{kr}} \right) \right) \\ &= \sum_{l=1}^n \sum_{k=1}^n V_{il} \bar{V}_{sk} E(Z_{lj} \bar{Z}_{kr}) = \delta_{jr} \sum_{l=1}^n V_{il} \bar{V}_{sl} = \delta_{jr} \delta_{is}. \end{aligned}$$

The  $i$ th column of  $VU$  is exactly  $VU_i$  and we have

$$VU_i = \frac{VZ_i - \sum_{l=1}^{i-1} \langle Z_i, U_l \rangle VU_l}{\left\| Z_i - \sum_{l=1}^{i-1} \langle Z_i, U_l \rangle U_l \right\|} = \frac{VZ_i - \sum_{l=1}^{i-1} \langle VZ_i, VU_l \rangle VU_l}{\left\| VZ_i - \sum_{l=1}^{i-1} \langle VZ_i, VU_l \rangle VU_l \right\|} \quad (4)$$

which is the Gram-Schmidt orthogonalization of the vectors  $VZ_i$ . Since we showed above that  $Z$  and  $VZ$  are identically distributed, we conclude that  $U$  and  $VU$  are identically distributed as well. Since the left invariance characterizes the Haar measure on a compact group, the above constructed  $U$  is Haar distributed and its distribution is right invariant as well.  $\square$

The column vectors of a unitary matrix are pairwise orthogonal unit vectors. On the bases of this fact we can determine a Haar unitary in a slightly different way. The complex unit vectors form a compact space on which the unitary group acts transitively. Therefore, there exist a unique probability measure invariant under the action. Let us call this measure uniform. To determine a Haar unitary, we choose the first

column vector  $U_1$  uniformly from the space of  $n$ -vectors.  $U_2$  should be taken from the  $n-1$  dimensional subspace orthogonal to  $U_1$  and choose it uniformly again. In general, if already  $U_1, U_2, \dots, U_j$  is chosen, we take  $U_{j+1}$  from the  $n-j$  dimensional subspace orthogonal to  $U_1, U_2, \dots, U_j$ , again uniformly. The column vectors constitute a unitary matrix and we check that its distribution is left invariant. Let  $V$  be a fixed unitary. We show that the vectors  $VU_1, VU_2, \dots, VU_n$  are produced by the above described procedure. They are obviously pairwise orthogonal unit vectors.  $VU_1$  is uniformly distributed by the invariance property of the distribution of  $U_1$ . Let  $V(1)$  be such a unitary that  $V(1)VU_1 = VU_1$ . Then  $V^{-1}V(1)VU_1 = U_1$  and the choice of  $U_2$  gives that  $V^{-1}V(1)VU_2 \sim U_2$ . It follows that  $V(1)VU_2 \sim VU_2$ . Since  $V(1)$  was arbitrary  $VU_2$  is uniformly distributed in the subspace orthogonal to  $VU_1$ . Similar argument works for  $VU_3, \dots, VU_n$ . The Gram-Schmidt orthogonalization of the columns of a Gaussian matrix gives a concrete realization of this procedure.

The permutation matrices are in  $\mathcal{U}(n)$ , and by multiplying with an appropriate permutation matrix every row and column can be transformed to any other row or column, so the translation invariance of a Haar unitary  $U$  implies that all the entries have the same distribution. From the above construction of a Haar unitary one can deduce easily the distribution of the entries:

$$\frac{n-1}{\pi}(1-r^2)^{n-2}r dr d\theta$$

(see also p. 140 in [6]). Since

$$P(|\sqrt{n}U_{ij}|^2 \geq x) = \left(1 - \frac{x}{n}\right)^{n-1} \rightarrow e^{-x}$$

$\sqrt{n}U_{ij}$  converges to a standard complex normal variable. The correlation coefficient between  $|U_{ii}|^2$  and  $|U_{jj}|^2$  is  $1/(n-1)^2$  if  $i \neq j$  (see p. 139 in [6]).

In the next section we need the following technical lemma which tells us that the expectation of many product of the entries are vanishing.

**Lemma 2.3** ([6]) *Let  $i_1, \dots, i_h, j_1, \dots, j_h \in \{1, \dots, n\}$  and  $k_1, \dots, k_h, m_1, \dots, m_h$  be positive integers for some  $h \in \mathbb{N}$ . If*

$$\sum_{i_r=u} (k_r - m_r) \neq 0 \quad \text{for some} \quad 1 \leq u \leq n$$

or

$$\sum_{j_r=v} (k_r - m_r) \neq 0 \quad \text{for some} \quad 1 \leq v \leq n, ,$$

then

$$E \left( (U_{i_1 j_1}^{k_1} \overline{U}_{i_1 j_1}^{m_1}) \dots (U_{i_h j_h}^{k_h} \overline{U}_{i_h j_h}^{m_h}) \right) = 0.$$

*Proof.* Suppose that  $t := \sum_{i_r=u} (k_r - m_r) \neq 0$ . The translation invariance of  $U$  implies that multiplying this matrix by  $V = \text{Diag}(1, \dots, 1, e^{i\theta}, 1, \dots, 1) \in \mathcal{U}(n)$  from the left we get

$$E \left( (U_{i_1 j_1}^{k_1} \overline{U}_{i_1 j_1}^{m_1}) \dots (U_{i_h j_h}^{k_h} \overline{U}_{i_h j_h}^{m_h}) \right) = e^{it\theta} E \left( (U_{i_1 j_1}^{k_1} \overline{U}_{i_1 j_1}^{m_1}) \dots (U_{i_h j_h}^{k_h} \overline{U}_{i_h j_h}^{m_h}) \right),$$

for all  $\theta \in \mathbb{R}$ .  $\square$

Let  $U$  be a Haar distributed  $n \times n$  unitary matrix with eigenvalues  $\lambda_0, \lambda_1, \dots, \lambda_{n-1}$ . The eigenvalues are random variables with values in  $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ , their joint distribution is well-known:

$$\frac{1}{n!} \prod_{i < j} |z_i - z_j|^2 = \frac{1}{n!} \prod_{i < j} |e^{i\theta_i} - e^{i\theta_j}|^2 \quad (5)$$

with respect to  $dz_0 dz_1 \dots dz_{n-1}$ , where  $dz_i = d\theta_i/2\pi$  for  $z_i = e^{i\theta_i}$  (see p. 135 in [6]).

**Theorem 2.4** *For  $m > n$  the random variables  $\lambda_0^m, \lambda_1^m, \dots, \lambda_{n-1}^m$  are independent and uniformly distributed on  $\mathbb{T}$ .*

*Proof.* Since the Fourier transform determines the joint distribution measure of  $\lambda_0^m, \lambda_1^m, \dots, \lambda_{n-1}^m$  uniquely, it suffices to show that

$$\int z_0^{k_0 m} z_1^{k_1 m} \dots z_{n-1}^{k_{n-1} m} \prod_{i < j} |z_i - z_j|^2 dz = 0 \quad (6)$$

if at least one  $k_j \in \mathbb{Z}$  is different from 0 ( $dz = dz_0 dz_1 \dots dz_{n-1}$  and integration is over  $\mathbb{T}^n$ ).

Let

$$\Delta(z_0, z_1, \dots, z_{n-1}) := \prod_{i < j} (z_i - z_j) = \det[z_i^k]_{0 \leq i \leq n-1, 0 \leq k \leq n-1}. \quad (7)$$

(What we have here is the so-called Vandermonde determinant.) Then

$$\prod_{i < j} |z_i - z_j|^2 = \Delta(z_0, z_1, \dots, z_{n-1}) \Delta(z_0^{-1}, z_1^{-1}, \dots, z_{n-1}^{-1})$$

and one can write (6) as an  $n$ -times complex contour integral along the positively oriented  $\mathbb{T}$ :

$$\begin{aligned} & \int z_0^{k_0 m} z_1^{k_1 m} \dots z_{n-1}^{k_{n-1} m} \Delta(z_0, z_1, \dots, z_{n-1}) \Delta(z_0^{-1}, z_1^{-1}, \dots, z_{n-1}^{-1}) dz \\ &= \int z_0^{k_0 m} \dots z_{n-1}^{k_{n-1} m} \sum_{\pi \in S_n} (-1)^{\sigma(\pi)} z_0^{\pi(0)} \dots z_{n-1}^{\pi(n-1)} \sum_{\rho \in S_n} (-1)^{\sigma(\rho)} z_0^{-\rho(0)} \dots z_{n-1}^{-\rho(n-1)} dz \end{aligned}$$

which is calculated according to the theorem of residue. We need to find the coefficient of  $z_0^{-1} z_1^{-1} \dots z_{n-1}^{-1}$ , so we are looking for the permutations, for which  $k_j m + \pi(j) - \rho(j) = -1$  if  $0 \leq j \leq n-1$ , so  $k_j m = \rho(j) - \pi(j) - 1$ . Here  $|\rho(j) - \pi(j)| \leq n-1$ , and  $|k_j m| \geq m > n$ , if  $k_j \neq 0$ , so if at least one  $k_j \in \mathbb{Z}$  is different from 0, then there exists no solution. This proves the theorem.  $\square$

### 3 Asymptotics of the trace

Let  $U(n) = (U(n)_{ij}) : \Omega \rightarrow \mathcal{U}(n)$  be a Haar distributed unitary random matrix. In this section we are interested in the convergence of  $\text{Tr } U(n)$  as  $n \rightarrow \infty$ . Since the correlation between the diagonal entries decreases with  $n$ , one expects on the basis of the central limit theorem, that the limit of the trace has complex normal distribution. We prove this by the method of moments.

**Theorem 3.1** *Let  $U(n)$  be a sequence of  $n \times n$  Haar unitary random matrices. Then  $\text{Tr } U(n)$  converges in distribution to a standard complex normal random variable as  $n \rightarrow \infty$ .*

*Proof.* For the sake of simplicity we write  $U$  instead of  $U(n)$ . First we study the asymptotics of the moments

$$\begin{aligned} E((\text{Tr } U)^k (\overline{\text{Tr } U})^k) &= E\left(\left(\sum_{i=1}^n U_{ii}\right)^k \left(\sum_{j=1}^n \overline{U}_{jj}\right)^k\right) \\ &= \sum_{i_1, \dots, i_k=1}^n \sum_{j_1, \dots, j_k=1}^n E(U_{i_1 i_1} \dots U_{i_k i_k} \overline{U}_{j_1 j_1} \dots \overline{U}_{j_k j_k}), \end{aligned}$$

$k \in \mathbb{Z}^+$ . By Lemma 2.3 parts of the above sum are zero, we need to consider only those sets of indices  $\{i_1, \dots, i_k\}$  and  $\{j_1, \dots, j_k\}$  which coincide (with multiplicities). Look at a summand  $E(|U_{i_1 i_1}|^{2k_1} \dots |U_{i_r i_r}|^{2k_r})$ , where  $\sum_{l=1}^r k_l = k$ . From the Hölder inequality

$$E(|U_{i_1 j_1}|^{2k_1} \dots |U_{i_r j_r}|^{2k_r}) \leq \prod_{l=1}^r \sqrt[2^l]{E(|U_{i_l j_l}|^{2 \cdot 2^l k_l})} = \prod_{l=1}^r \binom{n + 2^l k_l - 1}{2^l k_l - 1}^{-1/2^l} = O(n^{-k}). \quad (8)$$

The number of those sets of indices, where among the numbers  $i_1, \dots, i_k$  there are at least two equal is at most

$$k! \binom{k}{2} n^{k-1} = O(n^{k-1}).$$

By (8) the order of magnitude of these factors is  $O(n^{-k})$ , so this part of the sum tends to zero as  $n \rightarrow \infty$ .

Next we assume that  $i_1, \dots, i_k$  are different. Since by translation invariance any row or column can be replaced by any other, we have

$$E(|U_{i_1 i_1}|^2 \dots |U_{i_k i_k}|^2) = E(|U_{11}|^2 \dots |U_{kk}|^2) =: M_k^n. \quad (9)$$

It is enough to determine this quantity and to count how many of these terms are in the trace.

The length of the row vectors of the unitary matrix is 1, hence

$$\sum_{i_1=1}^n \dots \sum_{i_k=1}^n E(|U_{i_1 1}|^2 \dots |U_{i_k k}|^2) = 1. \quad (10)$$

We divide the sum into two parts: the number of terms with different indices is  $n!/(n-k)!$ , and again the translation invariance implies that each of them equals to  $M_k^n$ , and we denote by  $\varepsilon_k^n$  the sum of the other terms. Therefore

$$\varepsilon_k^n = 1 - \frac{n!}{(n-k)!} M_k^n \leq k! \binom{k}{2} O(n^{-k}) \rightarrow 0,$$

and

$$M_k^n = \frac{(1 - \varepsilon_k^n)(n-k)!}{n!}.$$

Now we can count how many expectations of value  $M_k^n$  are there in the sum (8). We can fix the indices  $i_1, \dots, i_k$  in  $n!/(n-k)!$  ways, and we can permute them in  $k!$  ways to get the indices  $j_1, \dots, j_k$ . The obtained equation

$$\lim_{n \rightarrow \infty} E((\text{Tr } U_n)^k (\overline{\text{Tr } U_n})^k) = \lim_{n \rightarrow \infty} \frac{n!}{(n-k)!} k! \frac{(1 - \varepsilon_k^n)(n-k)!}{n!} = k!$$

finishes the proof.

For the mixed moments we have by Lemma 2.3

$$E((\text{Tr } U_n)^k (\overline{\text{Tr } U_n})^m) = 0 \quad (k \neq m),$$

and we have proven the convergence of all moments. The only thing is left to conclude the convergence in distribution is to show that the moments determine uniquely the limiting distribution (VIII. 6 in [4]). By the Carleman criterion (in VII. 3 of [4]) for a real valued random variable the moments  $\gamma_k$  determine the distribution uniquely if

$$\sum_{k \in \mathbb{N}} \gamma_{2k}^{-\frac{1}{k}} = \infty.$$

Although we have complex random variables, the distribution of the argument is uniform, and we can consider them as real valued random variables. The Stirling formula tells us

$$\sum_{k \in \mathbb{N}} (k!)^{-\frac{1}{k}} \geq \sum_{k \geq M} \left( \left( \frac{2k}{e} \right)^k \right)^{-\frac{1}{k}} = \frac{e}{2} \sum_{k \geq M} \frac{1}{k} = \infty.$$

for a large  $M \in \mathbb{N}$ , since  $\sqrt{2k\pi} \leq 2^k$ , if  $k \geq 2$ . □



## 4 Asymptotic behaviour of the traces of higher powers

The aim of this section is to study the trace of higher powers of a Haar unitary. This was done also by Diaconis and Shashahani in [3]. Here we use elementary methods.

**Theorem 4.1** *Let  $Z$  be standard complex normal distributed random variable, then for the sequence of  $U_n$   $n \times n$  Haar unitary random matrices  $\text{Tr } U_n^l \rightarrow \sqrt{l}Z$  in distribution.*

*Proof.* We use the method of moments again. Lemma 2.3 implies that we only have to take into consideration  $E \left( (\text{Tr } U_n^l)^k \left( \overline{\text{Tr } U_n^l} \right)^k \right)$ , for all  $k \in \mathbb{Z}^+$ .

$$\begin{aligned} & E \left( (\text{Tr } U_n^l)^k \left( \overline{\text{Tr } U_n^l} \right)^k \right) \\ &= E \left( \left( \sum_{i_1, \dots, i_l} U_{i_1 i_2} U_{i_2 i_3} \dots U_{i_{l-1} i_l} U_{i_l i_1} \right)^k \left( \sum_{j_1, \dots, j_l} \overline{U}_{j_1 j_2} \overline{U}_{j_2 j_3} \dots \overline{U}_{j_{l-1} j_l} \overline{U}_{j_l j_1} \right)^k \right) \\ &= \sum E \left( U_{i_1 i_2} \dots U_{i_l i_1} U_{i_{l+1} i_{l+2}} \dots U_{i_{2l} i_{l+1}} \dots U_{i_{l(k-1)+1} i_{l(k-1)+2}} \dots U_{i_{kl} i_{l(k-1)+1}} \right. \\ & \quad \times \overline{U}_{j_1 j_2} \dots \overline{U}_{j_l j_1} \overline{U}_{j_{l+1} j_{l+2}} \dots \overline{U}_{j_{2l} j_{l+1}} \dots \overline{U}_{j_{l(k-1)+1} j_{l(k-1)+2}} \dots \overline{U}_{j_{kl} j_{l(k-1)+1}} \left. \right), \end{aligned}$$

where the indices  $i_1, \dots, i_{kl}, j_1, \dots, j_{kl}$  run from 1 to  $n$ , and by Lemma 2.3 if the sets  $\{i_1, \dots, i_{kl}\}$  and  $\{j_1, \dots, j_{kl}\}$  are different, then the expectation of the product is zero. It follows from the Cauchy and Hölder inequalities, and (8), that

$$\begin{aligned} & \left| E \left( U_{i_1 i_2} \dots U_{i_{kl} i_{l(k-1)+1}} \overline{U}_{j_1 j_2} \dots \overline{U}_{j_{kl} j_{l(k-1)+1}} \right) \right| \\ & \leq E \left| U_{i_1 i_2} \dots U_{i_{kl} i_{l(k-1)+1}} \overline{U}_{j_1 j_2} \dots \overline{U}_{j_{kl} j_{l(k-1)+1}} \right| \\ & \leq \sqrt{E \left( |U_{i_1 i_2}|^2 \dots |U_{i_{kl} i_{l(k-1)+1}}|^2 |\overline{U}_{j_1 j_2}|^2 \dots |\overline{U}_{j_{kl} j_{l(k-1)+1}}|^2 \right)} \leq O(n^{-kl}). \end{aligned} \tag{11}$$

Again the number of the set of indices, where there exist at least two equal indices is at most  $O(n^{kl-1})$ , so the sum of the corresponding expectations tends to zero as  $n \rightarrow \infty$ . Suppose that all the indices are different. There exist  $\frac{n!}{(n-kl)!} (kl)! = O(n^{kl})$  of these kinds of index sets, and now we will prove, that most of the corresponding products have order of magnitude less than  $n^{-kl-1}$ . Consider for any  $0 \leq r \leq kl$

$$N_k^n(r) := E \left( |U_{12}|^2 |U_{23}|^2 \dots |U_{r1}|^2 U_{r+1, r+2} \dots U_{kl-1, kl} U_{kl, r+1} \overline{U}_{r+2, r+1} \dots \overline{U}_{r+1, kl} \right).$$

Note that  $N_k^n(kl) = N_k^n(kl-1) = M_{kl}^n$ , and if  $\{i_1, \dots, i_{kl}\} = \{j_1, \dots, j_{kl}\}$ , and all the indices are different, then the corresponding term equals to  $N_k^n(r)$  for some  $0 \leq r \leq kl$ . Using the orthogonality of the rows for  $0 \leq r \leq kl-2$

$$E \left( \sum_{j=1}^n |U_{12}|^2 |U_{23}|^2 \dots |U_{r1}|^2 U_{r+1, r+2} \dots U_{kl-1, j} U_{kl, r+1} \overline{U}_{r+2, r+1} \dots \overline{U}_{r+1, j} \right) = 0. \tag{12}$$

If  $j \geq kl$ , then the permutation invariance implies, that

$$E(|U_{12}|^2|U_{23}|^2 \dots |U_{r1}|^2 U_{r+1,r+2} \dots U_{kl-1,j} U_{kl,r+1} \overline{U}_{r+2,r+1} \dots \overline{U}_{r+1,j}) = N_k^n(r),$$

so we can write from (12)

$$(n-kl)N_k^n(r) = -E\left(\sum_{j=1}^{kl} |U_{12}|^2|U_{23}|^2 \dots |U_{r1}|^2 U_{r+1,r+2} \dots U_{kl-1,j} U_{kl,r+1} \overline{U}_{r+2,r+1} \dots \overline{U}_{r+1,j}\right).$$

On the right side there is a sum of  $kl$  numbers which are less than  $O(n^{-kl})$  because of (11), so this equation holds only if  $N_k^n(r) \leq O(n^{-kl-1})$ .

We have to compute the sum of the expectations

$$E\left(|U_{i_1 i_2}|^2 \dots |U_{i_l i_1}|^2 \dots |U_{i_{(k-1)l+1} i_{(k-1)l+2}}|^2 \dots |U_{i_{kl} i_{(k-1)l+1}}|^2\right) = M_{kl}^n.$$

Now we count the number of these summands, so first we fix the set of sequences of length  $l$   $I_{l,k} = \{(i_{(u-1)l+1}, \dots, i_{ul}), 1 \leq u \leq k\}$ , and we try to find the set  $J_{l,k} = \{(j_{(u-1)l+1}, \dots, j_{ul}), 1 \leq u \leq k\}$ , which gives  $M_{kl}^n$ . If the product contains  $U_{i_r i_{r+1}}$ , then it has to contain  $\overline{U}_{i_r i_{r+1}}$ , so if  $i_r$  and  $i_{r+1}$  are in the same sequence of  $I_{l,k}$ , then  $j_s = i_r$  and  $j_t = i_{r+1}$  have to be in the same sequence of  $J_{l,k}$ , and  $t = s + 1$  modulo  $l$ . This means, that for all  $1 \leq u \leq k$  there exists a sequence  $(i_{(v-1)l+1}, \dots, i_{vl}) \in I_{k,l}$  and a cyclic permutation  $\pi$  of the numbers  $\{(v-1)l+1, \dots, vl\}$  such that  $(j_{(u-1)l+1}, \dots, j_{ul}) = (i_{\pi((v-1)l+1)}, \dots, i_{\pi(vl)})$ . We conclude, that for each  $I_{l,k}$  there are  $k!l^k$   $J_{l,k}$ , since we can permute the sets of  $I_{l,k}$  in  $k!$  ways, and in all sets there are  $l$  cyclic permutations. Clearly there are  $\frac{n!}{(n-kl)!}$  sets  $I_{l,k}$ , so

$$\lim_{n \rightarrow \infty} E\left((\text{Tr } U_n^l)^k (\overline{\text{Tr } U_n^l})^k\right) = \lim_{n \rightarrow \infty} \frac{n!}{(n-kl)!} k!l^k \frac{(1 - \varepsilon_{kl}^n)(n-kl)!}{n!} = k!l^k,$$

and as in the proof of Theorem 3.1 this is the  $k$ th moment of  $(\sqrt{l}Z)(\overline{\sqrt{l}Z})$ .  $\square$

## 5 Independence of the trace of the powers

In this section we prove that the limits of the trace of different powers are independent. The method of computation is the same as in the previous sections.

**Theorem 5.1** *Let  $U_n$  be a sequence of Haar unitary random matrices as above. Then  $\text{Tr } U_n, \text{Tr } U_n^2, \dots, \text{Tr } U_n^l$  are asymptotically independent.*

*Proof.* We will show, that the joint moments of  $\text{Tr } U_n, \text{Tr } U_n^2, \dots, \text{Tr } U_n^l$  converge to the joint moments of  $Z_1, \sqrt{2}Z_2, \dots, \sqrt{l}Z_l$ , where  $Z_1, Z_2, \dots, Z_l$  are independent standard

complex normal random variables. The latter joint moments are

$$E \left( \prod_{i=1}^l i^{\frac{a_i+b_i}{2}} Z_i^{a_i} \overline{Z_i}^{b_i} \right) = \prod_{i=1}^l i^{\frac{a_i+b_i}{2}} E \left( Z_i^{a_i} \overline{Z_i}^{b_i} \right) = \prod_{i=1}^l \delta_{a_i b_i} a_i! i^{a_i}.$$

From Lemma 2.3, if  $\sum_{i=1}^l i a_i \neq \sum_{i=1}^l i b_i$ , then the moment

$$E \left( \prod_{i=1}^l (\text{Tr } U_n^i)^{a_i} (\overline{\text{Tr } U_n^i})^{b_i} \right) = 0.$$

Now  $\sum i a_i = \sum i b_i$ . Again if the indices in the first and the second part are not the same, then the expectation is zero according to Lemma 2.3. The order of magnitude of each summand is at most  $O(n^{-\sum i a_i})$ , as above, so if not all the indices are different, then the sum of these expectations tends to zero, as  $n \rightarrow \infty$ . The same way as in the proof of the previous theorem, those summands where there is a  $U_{i_r i_{r+1}} \overline{U_{i_r i_s}}$ ,  $i_{r+1} \neq i_s$  in the product are small. So now we have to sum the expectations  $M_{\sum i a_i}^n$ . If we fix the set of first indices  $I$ , then again the sequences of the appropriate  $J$ , have to be cyclic permutations of the sequences of  $I$ . So the number of the sequences of length  $i$  in  $I$  is the same as in  $J$ , which means  $a_i = b_i$  for all  $1 \leq i \leq l$ . The number of the  $I$  sets is  $\frac{n!}{(n-\sum i a_i)!}$ , so we have arrived to

$$\begin{aligned} & \lim_{n \rightarrow \infty} E \left( \prod_{i=1}^l (\text{Tr } U_n^i)^{a_i} (\overline{\text{Tr } U_n^i})^{b_i} \right) \\ &= \lim_{n \rightarrow \infty} \frac{n!}{(n-\sum i a_i)!} \prod_{i=1}^l \delta_{a_i, b_i} i^{a_i} a_i! \frac{(1 - \varepsilon_{\sum i a_i}^n) (n - \sum i a_i)!}{n!} = \prod_{i=1}^l \delta_{a_i, b_i} a_i! i^{a_i}. \end{aligned}$$

□

## 6 Truncation

Let  $U$  be an  $n \times n$  Haar distributed unitary matrix. By truncating  $n - m$  bottom rows and  $n - m$  last columns, we get an  $m \times m$  matrix  $U_{[n, m]}$ . In this section we study the limit of  $U_{[n, m]}$  when  $n \rightarrow \infty$  and  $m$  is fixed. Our method is based on the explicit form of the joint eigenvalue density.

The truncated matrix is not unitary but it is a contraction. Hence the eigenvalues  $z_1, z_2, \dots, z_m \in D^m$ , where  $D = \{z \in \mathbb{C} : |z| \leq 1\}$  is the unit disc. According to [9] the joint probability density of the eigenvalues is

$$C_{[n, m]} \prod_{i < j} |\zeta_i - \zeta_j|^2 \prod_{i=1}^m (1 - |\zeta_i|^2)^{n-m-1}$$

on  $D^m$ .

Since the normalizing constant  $C_{[n,m]}$  was not given in [9], we first compute it by integration. To do this, we write  $\zeta_i = r_i e^{i\varphi_i}$  and  $d\zeta_i = r_i dr_i d\varphi_i$ . Then

$$\begin{aligned} C_{[n,m]}^{-1} &= \int_{D^m} \prod_{1 \leq i < j \leq m} |z_i - z_j|^2 \prod_{i=1}^m (1 - |z_i|^2)^{n-m-1} dz \\ &= \int_{[0,1]^m} \int_{[0,2\pi]^m} \prod_{1 \leq i < j \leq m} |r_i e^{i\varphi_i} - r_j e^{i\varphi_j}|^2 \prod_{i=1}^m (1 - r_i^2)^{n-m-1} \prod_{i=1}^m r_i d\varphi dr. \end{aligned}$$

Next we integrate with respect to  $d\varphi = d\varphi_1 d\varphi_2 \dots d\varphi_m$  by transformation into complex contour integral what we evaluate by means of the residue theorem.

$$\begin{aligned} &\int_{[0,2\pi]^n} \prod_{1 \leq i < j \leq m} |r_i e^{i\varphi_i} - r_j e^{i\varphi_j}|^2 d\varphi \\ &= (-i)^m \int_{\mathbb{T}^n} \prod_{1 \leq i < j \leq m} |r_i z_i - r_j z_j|^2 \prod_{i=1}^m z_i^{-1} dz \\ &= (-i)^m \int_{\mathbb{T}^n} \prod_{1 \leq i < j \leq m} (r_i z_i - r_j z_j)(r_i z_i^{-1} - r_j z_j^{-1}) \prod_{i=1}^m z_i^{-1} dz \\ &= (-i)^m \int_{\mathbb{T}^n} \prod_{i=1}^m z_i^{-1} \det \begin{bmatrix} 1 & 1 & \dots & 1 \\ r_1 z_1 & r_2 z_2 & \dots & r_m z_m \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{m-1} z_1^{m-1} & r_2^{m-1} z_2^{m-1} & \dots & r_m^{m-1} z_m^{m-1} \end{bmatrix} \times \\ &\quad \times \det \begin{bmatrix} 1 & 1 & \dots & 1 \\ r_1 z_1^{-1} & r_2 z_2^{-1} & \dots & r_m z_m^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{m-1} z_1^{-(m-1)} & r_2^{m-1} z_2^{-(m-1)} & \dots & r_m^{m-1} z_m^{-(m-1)} \end{bmatrix} dz \\ &= (-i)^m \int_{\mathbb{T}^n} \prod_{i=1}^m z_i^{-1} \sum_{\pi \in S_m} (-1)^{\sigma(\pi)} \prod_{i=1}^m (r_i z_i)^{\pi(i)-1} \sum_{\rho \in S_m} (-1)^{\sigma(\rho)} \prod_{i=1}^m (r_i z_i^{-1})^{\rho(i)-1} dz. \end{aligned}$$

We have to find the coefficient of  $\prod_{i=1}^m z_i^{-1}$ , this gives that only  $\rho = \pi$  contribute and the integral is

$$(2\pi)^m \sum_{\rho \in S_m} \prod_{i=1}^m (r_i)^{2(\rho(i)-1)}.$$

So we have

$$\begin{aligned} C_{[n,m]}^{-1} &= (2\pi)^m \int_{[0,1]^m} \sum_{\rho \in S_m} \prod_{i=1}^m (r_i)^{2(\rho(i)-1)} \prod_{i=1}^m (1 - r_i^2)^{n-m-1} \prod_{i=1}^m r_i dr \\ &= (2\pi)^m m! \prod_{i=1}^m \int_0^1 r_i^{2i-1} (1 - r_i^2)^{n-m-1} dr_i \end{aligned}$$

and the rest is done by integration by parts:

$$\begin{aligned}
\int_0^1 r^{2k+1}(1-r^2)^{n-m-1} dr &= \frac{k}{n-m} \int_0^1 r^{2k-1}(1-r^2)^{n-m} dr \\
&= \frac{k!}{(n-m)\dots(n-m+k-1)} \int_0^1 r(1-r^2)^{n-m+k-1} dr \\
&= \binom{n-m+k-1}{k}^{-1} \frac{1}{2(n-m+k)}.
\end{aligned}$$

Therefore

$$C_{[n,m]}^{-1} = \pi^m m! \prod_{k=0}^{m-1} \binom{n-m+k-1}{k}^{-1} \frac{1}{n-m+k}.$$

Now we consider  $\sqrt{n/m} U_{[n,m]}$ . Its joint probability density of the eigenvalues is simply derived from the above density of  $U_{[n,m]}$  by the transformation

$$(\zeta_1, \dots, \zeta_m) \mapsto \left( \sqrt{\frac{m}{n}} \zeta_1, \dots, \sqrt{\frac{m}{n}} \zeta_m \right),$$

and it is given as

$$\begin{aligned}
C_{[n,m]} &\left( \frac{m}{n} \right)^m \prod_{i < j} \left| \sqrt{\frac{m}{n}} \zeta_i - \sqrt{\frac{m}{n}} \zeta_j \right|^2 \prod_{i=1}^m \left( 1 - \frac{m|\zeta_i|^2}{n} \right)^{n-m-1} \\
&= \frac{1}{\pi^m m!} \prod_{k=0}^{m-1} \binom{n-m+k-1}{k} (n-m+k) \left( \frac{m}{n} \right)^{m(m+1)/2} \\
&\quad \times \prod_{i < j} |\zeta_i - \zeta_j|^2 \prod_{i=1}^m \left( 1 - \frac{m|\zeta_i|^2}{n} \right)^{n-m-1} \\
&= \frac{1}{\pi^m m!} \prod_{k=0}^{m-1} \frac{n^{k+1}(1+o(1))}{k!} \left( \frac{m}{n} \right)^{m(m+1)/2} \prod_{i < j} |\zeta_i - \zeta_j|^2 \prod_{i=1}^m \left( 1 - \frac{m|\zeta_i|^2}{n} \right)^{n-m-1} \\
&= \frac{m^{m(m+1)/2}}{\pi^m \prod_{k=1}^m k!} (1+o(1)) \prod_{i < j} |\zeta_i - \zeta_j|^2 \prod_{i=1}^m \left( 1 - \frac{m|\zeta_i|^2}{n} \right)^{n-m-1}.
\end{aligned}$$

The limit of the above as  $n \rightarrow \infty$  is

$$\frac{m^{m(m+1)/2}}{\pi^m \prod_{k=1}^m k!} \exp \left( -m \sum_{i=1}^m |\zeta_i|^2 \right) \prod_{i < j} |\zeta_i - \zeta_j|^2,$$

which is exactly the joint eigenvalue density of the standard  $m \times m$  non-selfadjoint Gaussian matrix. This implies the following theorem.

**Theorem 6.1** *The normalized truncated matrix*

$$\sqrt{\frac{n}{m}}U_{[n,m]}$$

*converge in distribution to the standard  $m \times m$  non-selfadjoint Gaussian matrix.*

Details of this convergence will be subject of a forthcoming publication.

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